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Mathieu Functions of General Order: Connection Formulae, Base Functions and Asymptotic Formulae: I. Introduction

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MATHIEU FUNCTIONS OF GENERAL ORDER: CONNECTION FORMULAE, BASE FUNCTIONS AND ASYMPTOTIC FORMULAE

I. INTRODUCTION

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CONTENTS

This is the first, introductory, paper of a series devoted to the derivation of a comprehensive set of approximate formulae for solutions of Mathieu's equation with real parameters, in terms both of elementary and of higher transcendental functions. Order-of-magnitude error-estimates are obtained; these in every case reflect faithfully the behaviour of the actual error over the appropriate range of parameters and of independent variable.

The general scope of the work is outlined in this Introduction, and is compared with that of previous work, in particular that of Langer (1934*b*). There then follows a description of the plan of the work and of the content of the several parts.

1. Antecedents

Several studies have been made of the problem of constructing approximations for solutions of the Mathieu equation

$$y'' + (\lambda - 2q \cos 2x)\, y = 0.$$

The most comprehensive of these appears to be that of Langer (1934*b*). Langer uses the method of the Liouville–Green (L.–G. or W.K.B.) approximation, with certain extensions and refinements developed by him (Langer 1934*a*), including a procedure for estimating the error of the approximations. Since then, various authors have further extended and to a considerable extent reformulated these methods; for a detailed account, reference may be made to Olver (1974), who not only brings together much of his own work in this field, but gives historical notes and references to other work.

In Langer's (1934*b*) paper, denoted subsequently by 'L.', mention is made of other types of approximation including both convergent and asymptotic series of restricted validity; for these, as well as for basic properties of solutions of the Mathieu equation, reference may be made to standard works on Mathieu functions, for example Meixner & Schäfke (1954), which contains an extensive bibliography, McLachlan (1947) and Arscott (1964). Brief accounts of Mathieu functions also appear in several texts on mathematical methods, in particular Morse & Feshbach (1953). Further asymptotic expansions have been obtained by Blanch (1960), Dingle & Müller (1962, 1964) and Jorna (1965), but as usual for such expansions relating to Mathieu functions, the treatment appears to be purely formal.

The case where the parameters are complex is treated by Sharples (1967, 1971), but in other respects the parameters are considerably restricted. It is unfortunate that there are errors in these two papers, relating in particular to the identification of solutions and to their periodic properties. With the notation of Sharples (1967), the solutions $W_j(x)$ of the modified Mathieu equation, which are characterized by their asymptotic behaviour as $x \to \pm\infty$, are identified with multiples of $M_\nu^{(3)}(z)$ or of $M_\nu^{(4)}(z)$ (Meixner & Schäfke 1954); in certain cases the choice of superscript is wrong and in others the identification should be with one of $M_\nu^{(3)}(-z)$ or $M_\nu^{(4)}(-z)$. Also, periodicity is attributed to these solutions; this is always false except in the case where every solution of the ordinary Mathieu equation has period 4π.

2. The scope of the present series

This is most easily described by means of a comparison with L., which therefore follows, in points (*a*)–(*e*).

(*a*) As in L., the parameters are restricted to real values while the variable is complex; otherwise the only requirement is that *either* $|q|$ should be large *or* λ should be bounded away from zero with $|\lambda/q|$ large, when q may be small.

(*b*) Error estimates in L. are all in the form of 'uniform estimates'; more precisely, the estimate of the error relative to a suitable majorant function for the principal term in the approximation is uniform over the whole domain of the variable involved, though it depends on the parameters.

The relative remainder estimates obtained below also contain in effect an unspecified bounding constant, but they reflect much more closely the actual behaviour of the remainder. They are all of the form $\omega(z, \lambda, q)\ O(1)$, where the first factor is an explicit function of the independent variable and parameters, while the factor $O(1)$ represents a function with the same arguments which is *bounded* over the domain of (z, λ, q) specified in each case. To obtain estimates of this form involves considerable extra calculation, but extends the range of validity of the formulae and has greater accuracy under certain conditions. In fact, the validity of these estimates requires only that the explicit factor be subject to an arbitrary (but fixed) bound; it does not depend directly on λ or q being large. As an illustration, for the Bessel function approximations of part V, §4.4, the relative remainder is at least as small as $\lambda^{-\frac{3}{2}} q\ O(1)$ on the half-strip $\{z: 0 \leqslant \mathrm{Re}\ z \leqslant \frac{1}{2}\pi,\ \mathrm{Im}\ z \geqslant 0\}$ provided only that $|\lambda/q| \geqslant 4$ and that λ^{-1} is bounded; subject to these conditions this result remains valid even if q tends to zero.

(*c*) L. only gives formulae for odd and even functions, both for $q > 0$ and for $q < 0$, and such a pair does not in general form a satisfactory basis for the system of solutions; the existence of pairs that do form such a basis is indicated, but the question is not pursued. Again, the dependence of the qualitative behaviour of the odd and even solutions on the parameters is implicit in L.'s formulae, although the errors can, for $q < 0$, be considerably larger than it might appear, because of a certain ambiguity in the notation for representing the error. Solutions of other types, such as Floquet solutions, solutions of the third kind and modified functions of the second kind, and their relation with the odd and even solutions, are not considered.

In the present paper, on the other hand, such relations are comprehensively treated, all being expressed in terms of a single pair of *precisely defined* auxiliary parameters, functions of (λ, q); asymptotic formulae for these auxiliary parameters are naturally included. For real-variable solutions, both of the ordinary and of the modified Mathieu equation, the construction of pairs of base functions satisfying criteria introduced by Miller (1950) in connection with Airy functions, together with these relations, facilitate, for example, the examination of the connection between the qualitative behaviour of particular solutions and the characteristic exponent. These criteria concern the phase relation between the two solutions when they are oscillatory, and the growth-rate of their ratio when they are hyperbolic in behaviour.

(*d*) Different cases – 'configurations' – are treated in L. as referring to mutually exclusive parameter ranges, each with a different type of approximation, though a measure of overlap can be obtained by choosing different values for certain unspecified bounds. Here, however, each method is applied over the widest possible parameter range, except where for computational convenience certain precise bounds have been specified when this was not in fact necessary, specifically, the values $|\lambda/q| = 4$ and $\lambda = 0$ in connection with approximations in terms of parabolic cylinder functions and Bessel functions.

Similarly, while in L. different formulae are given in a neighbourhood of a transition point and elsewhere, here most formulae obtained are applicable on the whole of the relevant domain, except at a transition point in the case of elementary function approximations.

(*e*) L. does not refer to Airy functions, though his formulae in terms of Bessel functions in the neighbourhood of a transition point are equivalent to Airy function approximations. The latter functions appear to have been first introduced into problems of this kind by Jeffreys (1942).

The approximations given in L. in terms of Whittaker functions can readily be expressed in terms of parabolic cylinder functions, which seem more appropriate, since the transformation of independent variables involved is then one-to-one on a region containing the two relevant transition points. The method used here for obtaining such approximations is different from that in L. and depends on a Liouville transformation which appears to have been first introduced by Kazarinoff (1958), who treats a differential equation with a fixed pair of simple transition points. Olver (1975) uses the same transformation, and obtains general results in the real-variable case when the transition points are variable and may coalesce. In part V, this transformation is used in an *ad hoc* manner, adapted to the particular application. No general theory of the complex-variable case with coalescent turning points is developed; it does not seem that to do so would shorten the analysis significantly.

The range of validity of the formulae in L. is smaller, but the argument of the Whittaker functions in L. does in fact provide an approximation to that of the parabolic cylinder functions in the present paper. Approximations in terms of parabolic cylinder functions are also given in Hansen (1962); these, however, are not asymptotic as $\operatorname{Im} x \to \infty$.

Finally, for the parameter ranges in which $|\lambda/q|$ is large, solutions are obtained in L. only on a domain on which $|\operatorname{Im} x|$ is bounded. Here, there is no such limitation; approximations are obtained on unbounded domains, both in terms of elementary functions and of Bessel functions, the latter being uniformly valid in the neighbourhood of a pair of transition points, and also as $q \to 0$ provided only that λ is bounded away from zero.

3. The plan of the present series

Part II examines connection formulae relating a solution $y(x)$ of the Mathieu equation and the solutions $y(\pm x \pm n\pi)$ generated from it by the fundamental group of the equation. The treatment is exact and is made first in the context of more general periodic differential equations; the results are then specialized to the Mathieu equation, a function of the third kind, characterized by its asymptotic behaviour as $x \to \infty i$, being taken as fundamental.

As in L., the parameters (λ, q) are taken as basic, rather than q and the characteristic exponent $\pi\nu$, which may appear inconvenient for many applications. The reason for this choice is that the solutions behave in a strongly irregular manner as the exponent varies; indeed λ is not uniquely determined by $(\pi\nu, q)$ unless ν is either real or purely imaginary, with the conventional determination of the real part.

At this point, it becomes necessary to distinguish two different parameter ranges, corresponding to the regions of the stability diagram where the solutions are always unstable, and where subregions of stability and instability alternate. The auxiliary parameters referred to above are then defined, differently in the two cases, and appropriate pairs of real-variable base functions are constructed, solutions of the two types of modified equation corresponding respectively to $q > 0$ and to $q < 0$, and of the ordinary equation, where a change of sign of q corresponds to a displacement of x by $\frac{1}{2}\pi$.

Part III comprises an account of the L.–G. method, with estimation of the error, and of its generalization to the determination of formulae in terms of the solutions of some specified 'basic equation' such as the Airy equation. Special attention is given to the basic equations that are used in parts IV and V, and there are sections devoted to the identification of solutions, to the estimation of connection coefficients and to the computational aspects of the estimation of the remainder; asymptotic series developments are not considered. There is little in part III that is essentially new, though some aspects of the presentation are thought to be novel. In particular, a feature that is emphasized is the close connection between each of the independent variables used for formulae in terms of different higher functions and that used for elementary-function approximations.

Part IV is devoted to the application of the methods described in part III, together with the formulae of part II, to the derivation of various asymptotic formulae in terms of exponential and circular functions. Note that by a 'transition point' of a differential equation of the form $y'' = F(x)\,y$ is meant a point where the coefficient $F(x)$ vanishes or, possibly, has a pole; in the real-variable case, a simple zero of $F(x)$ corresponds to a transition from exponential to oscillatory behaviour of the solutions. Thus, elementary-function approximations are not valid in the neighbourhood of any transition point, while approximations in terms of relatively simple higher functions can be found which are valid in the neighbourhood of one or more such points.

The possibilities are set out more fully in the introductory section of part V, and in the following sections approximations in terms of the various types of higher function are obtained; in each case, the formulae are collected together in tabular form at the end of the section. Parts II and III conform to the same pattern, the formulae being collected in the last section of the part.

4. Conclusion

There are two natural directions that might be followed in the application of similar methods to a further study of approximations for Mathieu functions. One would be to refine the procedure for estimating the error-control functions sufficiently to lead to realistic upper bounds for the various remainder terms, in place of mere order-of-magnitude estimates; this problem the author has so far found quite intractable. The other, which does not seem essentially difficult, is to extend those results for which it is appropriate, to the case where the parameters λ, q take general complex values. Some exploratory work in this direction has led to interesting tentative results relating to the Riemann surface of the characteristic values (i.e. values for which there exist periodic solutions of the differential equation) of λ as a function of q. Not surprisingly it appears to consist of four connected components, with infinitely many branch points, corresponding to the four classes of periodic Mathieu functions, ce_{2n}, se_{2n}, ce_{2n+1}, se_{2n+1}. In addition, the complete structure of the surface emerges, and estimates are obtainable for the radius of convergence of the Taylor series for a_n, b_n in powers of q: it turns out that for large n, the radius is of the order of n^2, whereas the lower bound given in Meixner & Schäfke (1954, ch. 2, theorem 7) is of the order of n only.

Note added in proof, 14 *January* 1981. The author has learnt that some results concerning the branch-points of a_n, b_n as functions of q will appear shortly in Hunter & Guerrieri (1981).

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